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USE OF THE REYNOLDS FLUX CONCEPT FOR ANALYSING ONE-DIMENSIONAL TWO-PHASE FLOW

PART II. APPLICATIONS TO TWO-PHASE FLOW

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NOMENCLATURE

(Same as in Part I, with the following modifications and additions).

a,	any thermodynamic property;
A,	cross-section area;
C,	a constant [equation (53)];
D,	diameter;
F,	force;
f_w,	wall friction factor;
G,	mass velocity;
g,	acceleration due to gravity;
h,	enthalpy;
M,	mass exchange per unit volume per unit time;
M_f,	pseudo-Mach number for liquid;
M_g,	pseudo-Mach number for vapor;
m_{fg},	vaporisation flux;
p,	pressure;
P_i,	interface perimeter;
$\frac{dq}{dt}$,	rate of heat addition per unit length;
s,	entropy;
S_s,	source of entropy per unit length;
W,	mass flow rate;
x,	vapor quality;
dy,	element of duct perimeter;
z,	coordinate in direction of flow.

Greek symbols

α,	void fraction;
δ,	film thickness;
γ,	ratio between mean square velocity and square of mean velocity;
θ,	inclination to the vertical;
λ,	a mathematical device.

Subscripts

f,	liquid;
g,	vapor;
i,	interface;
1, 2,	at locations 1 and 2 in the flow.

INTRODUCTION

THE PREVIOUS paper has served to establish the usefulness of the Reynolds flux concept for analysing mass-transfer processes. In particular, simple expressions were developed for describing the interaction between the mixing, or recirculating, flux and a superimposed net mass flux.

In this paper these techniques will be used to predict pressure gradients in two-phase flow. It will be shown that the Reynolds flux provides a unifying concept which enables several analytical models to be viewed as special, or limiting, cases of a general set of equations. Moreover, the terms in these equations will be related to the several mechanisms of entropy generation by friction.

Although a variety of applications will be presented in this paper, it seems likely that other useful results could be achieved using similar methods. For example, the problem of the critical heat flux in forced convection may well be amenable to analysis as a process of counter-current flow of liquid and vapor normal to the heated surface, similar to Fig. 1 of the previous paper [1] and reminiscent of the well-known Zuber [2] derivation for the case of pool boiling. Recently, Kutateladze and Leontiev [3] have successfully correlated highly subcooled critical heat flux data in terms of a model of "boundary-layer separation" due to a high value of material flux from the heated wall as a result of vapor formation. This is equivalent to the condition that ε_m should be zero. One might hope that a suitably modified theory would be valid for finite values of ε_m and would provide an explanation of these phenomena at low sub-coolings.

A significant contribution of the present theory is its ability to predict pressure gradients in the presence of condensation with greater accuracy than previous techniques.

GENERAL ONE-DIMENSIONAL TWO-PHASE (LIQUID–VAPOR) FLOW EQUATIONS

Consider the one-dimensional steady flow of a vapor–liquid mixture as illustrated in Fig. 1. Let the mass flow rates of the components be W_f and W_g and let the fraction of the cross-section area which is occupied by the vapor be α. Let there be a mass flux m due to vaporization over an interfacial perimeter P_i and a wall shear stress τ_w acting on elements of pipe perimeter dy_f and dy_g for each component. If the liquid is regarded as having a uniform velocity v_f and there is a recirculating Reynolds flux ε_m in the vapor stream, the momentum equations for the separate phases over a differential length dz, are

$$W_g v_g + mP_i v_f \, dz = (W_g + mP_i \, dz)(v_g + dv_g) + P_i \varepsilon_m (v_g - v_f) \, dz + \alpha A \, dp + dz \int_g \tau_w \, dy_g - \alpha A \rho_g g \cos\theta \qquad (1)$$

$$W_f v_f = (W_f - mP_i \, dz)(v_f + dv_f) - P_i \varepsilon_m (v_g - v_f) \, dz + (1 - \alpha) A \, dp + mP_i v_f + dz \int_f \tau_w \, dy_f - (1 - \alpha) A \rho_f g \cos\theta. \qquad (2)$$

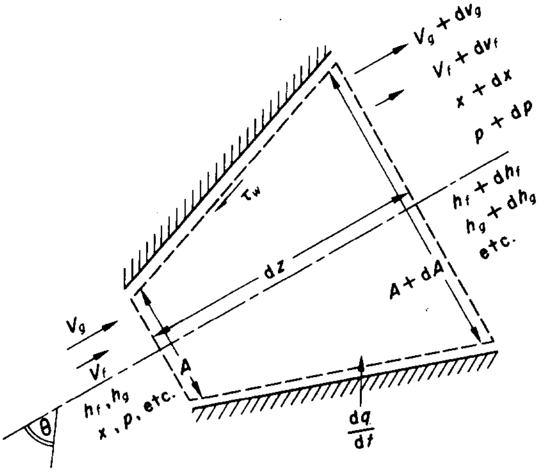

FIG. 1. Control volume for one-dimensional two-phase flow analysis.

The flow rates are related to the component velocities by the equations

$$W_g = \rho_g v_g \alpha A \qquad (3)$$

$$W_f = \rho_f v_f (1 - \alpha) A. \qquad (4)$$

Retaining only the first-order terms in equations (1) and (2) and making use of (3) and (4) we obtain eventually the following two equations of motion

$$\rho_g v_g \frac{dv_g}{dz} = -\frac{dp}{dz} - \rho_g g \cos\theta - \frac{1}{\alpha A} \int_g \tau_w \, dy_g - \frac{P_i}{A\alpha} (v_g - v_f)(\varepsilon_m + m) \qquad (5)$$

$$\rho_f v_f \frac{dv_f}{dz} = -\frac{dp}{dz} - \rho_f g \cos\theta - \frac{1}{(1 - \alpha) A} \times \int_f \tau_w \, dy_f - \frac{P_i}{A(1 - \alpha)} (v_f - v_g) \varepsilon_m. \qquad (6)$$

The recirculating Reynolds flux can be related to the flux without evaporation by using one of the equations from the previous paper. The simplest one is

$$\varepsilon_m = \varepsilon_o - \frac{m}{2}. \qquad (7)$$

When this value is substituted into the final terms in equations (5) and (6) the results are:

for the vapor

$$\frac{P_i}{\alpha A}(v_g - v_f)(\varepsilon_m + m) = \frac{P_i}{\alpha A}(v_g - v_f)\left(\varepsilon_o + \frac{m}{2}\right). \qquad (8)$$

for the liquid

$$\frac{P_i}{(1-\alpha)A}(v_f - v_g)\,\varepsilon_m = \frac{P_i}{(1-\alpha)A}(v_f - v_g) \times \left(\varepsilon_o - \frac{m}{2}\right). \qquad (9)$$

If condensation were taking place, instead of vaporisation, the sign of m would not change in (7) but there would be a factor $(\varepsilon_m + m)$ for the liquid and only ε_m for the vapor in equations (8) and (9). The result would be exactly the same as the effect of changing the algebraic sign of m in the right-hand side of equations (8) and (9). Denoting the interphase flux from liquid to vapor by the symbol m_{fg}, which can be either positive or negative, we can therefore rewrite equations (5) and (6) in a symmetrical form which is valid for both evaporation and condensation, as follows,

$$\rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} = -\frac{\mathrm{d}p}{\mathrm{d}z} - \rho_g g\cos\theta - \frac{1}{\alpha A}\int_g \tau_w\,\mathrm{d}y_g - \frac{P_i}{A\alpha}(v_g - v_f)\left(\varepsilon_o + \frac{m_{fg}}{2}\right) \qquad (10)$$

$$\rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} = -\frac{\mathrm{d}p}{\mathrm{d}z} - \rho_f g\cos\theta - \frac{1}{(1-\alpha)A} \times \int_f \tau_w\,\mathrm{d}y_f - \frac{P_i}{A(1-\alpha)}(v_f - v_g)\left(\varepsilon_o - \frac{m_{fg}}{2}\right). \qquad (11)$$

Most equations in the literature contain the implicit assumption that ε_m is equal to ε_o and is not affected by phase change. The resulting equations are not symmetrical and do not allow evaporation and condensation to be analysed on the same basis.

The above derivation loses some generality by focusing attention only on the Reynolds flux in the vapor stream. A more general analysis which ascribes a flux ε_{og} to the gas stream and a flux ε_{of} to the liquid stream and considers momentum transfer from both streams to an interface moving at v_i yields the result

$$v_i = \frac{v_g\left(\varepsilon_{og} - \frac{m_{fg}}{2}\right) + v_f\left(\varepsilon_{of} + \frac{m_{fg}}{2}\right)}{\varepsilon_{og} + \varepsilon_{of}}. \qquad (12)$$

The final terms in equations (10) and (11) are then

$$\frac{P_i}{A\alpha}\,\frac{(v_g - v_f)\left(\varepsilon_{og} + \frac{m_{fg}}{2}\right)\left(\varepsilon_{of} + \frac{m_{fg}}{2}\right)}{\varepsilon_{og} + \varepsilon_{of}}$$

and

$$\frac{P_i}{A(1-\alpha)}\,\frac{(v_f - v_g)\left(\varepsilon_{og} - \frac{m_{fg}}{2}\right)\left(\varepsilon_{of} - \frac{m_{fg}}{2}\right)}{\varepsilon_{og} + \varepsilon_{of}}.$$

In most cases ε_{of} is much greater than both ε_{og} and m_{fg} and these results reduce to equations (10) and (11).

If equation (10) is multiplied by α and equation (11) by $(1 - \alpha)$ and the results are added we obtain the "equation of motion of the mixture" which represents the momentum balance for the entire flow over the length dz, as follows

$$\alpha v_g \rho_g \frac{\mathrm{d}v_g}{\mathrm{d}z} + (1-\alpha)\rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} = -\frac{\mathrm{d}p}{\mathrm{d}z} - g\cos\theta[\alpha\rho_g + (1-\alpha)\rho_f] - \frac{1}{A}\int \tau_w\,\mathrm{d}y - \frac{P_i}{A}(v_g - v_f)\,m_{fg}. \qquad (13)$$

The final term in this equation represents the impulse which is given to the vaporising fluid per unit time, per unit volume. From the continuity relationships for the vapor and liquid, we have

$$\frac{\mathrm{d}W_g}{\mathrm{d}z} = -\frac{\mathrm{d}W_f}{\mathrm{d}z} = P_i\,m_{fg}. \qquad (14)$$

Usually the vapor and liquid flow rates are expressed in terms of the overall flow rate and the "quality", x, as follows

$$W_g = x\,W \tag{15}$$

$$W_f = (1 - x)\,W. \tag{16}$$

The mass velocity, G, is defined as

$$G = W/A. \tag{17}$$

Using equations (14) through (17) the last term in equation (13) can be rewritten as

$$\frac{P_i}{A}(v_g - v_f)\,m_{fg} = G(v_g - v_f)\frac{\mathrm{d}x}{\mathrm{d}z}. \tag{18}$$

Using (18) in (13) and making use of (3) and (4) an alternative form of the momentum equation for both phases taken together is found to be

$$G\frac{\mathrm{d}}{\mathrm{d}z}[x\,v_g + (1 - x)\,v_f] = -\frac{\mathrm{d}p}{\mathrm{d}z} - g\cos\theta[\alpha\,\rho_g + (1 - \alpha)\,\rho_f] - \frac{1}{A}\int \tau_w\,\mathrm{d}y. \tag{19}$$

An alternative approach via entropy generation

Equations (10) and (11) may be approached by an entirely different route which utilizes consideration of entropy generation.

First, write the energy equation for the control volume shown in Fig. 1, as follows

$$\frac{1}{W}\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}z}[x\,h_g + (1 - x)\,h_f] + \frac{\mathrm{d}}{\mathrm{d}z}\left[x\frac{v_g^2}{2} + (1 - x)\frac{v_f^2}{2}\right] + g\cos\theta. \tag{20}$$

The quantity $\mathrm{d}q/\mathrm{d}t$ denotes the rate of heat addition per unit length. Shaft and shear work terms are assumed to be zero.

An entropy balance may also be drawn up for the same control volume. If S_s denotes a source of entropy per unit length due to irreversible processes and heat transfer we have

$$S_s = W\frac{\mathrm{d}}{\mathrm{d}z}[x\,s_g + (1 - x)\,s_f]. \tag{21}$$

Now, from thermodynamic theory it is known that, for a homogeneous pure substance in equilibrium

$$T\,\mathrm{d}s = \mathrm{d}h - \frac{1}{\rho}\mathrm{d}p. \tag{22}$$

For a two-phase mixture, moreover

$$T\mathrm{d}[x\,s_g + (1 - x)\,s_f] = \mathrm{d}[x\,h_g + (1 - x)h_f] - \left[\frac{x}{\rho_g} + \frac{1 - x}{\rho_f}\right]\mathrm{d}p. \tag{23}$$

Although equation (23) was deduced for the particular case of a homogeneous mixture, it is an algebraic relationship between thermodynamic properties which is valid for any change of state for which the phases remain in equilibrium. Therefore we may divide equation (23) by dz, substitute from equations (19–21) and make use of (3, 4, 15, 16–17) to obtain eventually,

$$TS_s = \frac{\mathrm{d}q}{\mathrm{d}t} + \alpha(1 - \alpha)(v_g - v_f)A\left[g\cos\theta(\rho_f - \rho_g) + \frac{1}{2}\left\{G(v_g - v_f)\left(\frac{1}{1 - \alpha} - \frac{1}{\alpha}\right)\frac{\mathrm{d}x}{\mathrm{d}z} - \rho_g\frac{\mathrm{d}v_g^2}{\mathrm{d}z} + \rho_f\frac{\mathrm{d}v_f^2}{\mathrm{d}z}\right\}\right] + G\int \tau_w\,\mathrm{d}y\left[\frac{x}{\rho_g} + \frac{1 - x}{\rho_f}\right]. \tag{24}$$

Furthermore, the final term in equation (24) can be rearranged as follows, using (3, 4, 15, 16–17)

$$\begin{aligned} G\left[\frac{x}{\rho_g} + \frac{1 - x}{\rho_f}\right]\int_f \tau_w\,\mathrm{d}y &= \left\{\alpha v_g + (1 - \alpha)\,v_f\right\} \times \left\{\int_f \tau_w\,\mathrm{d}y_f + \int_g \tau_w\,\mathrm{d}y_g\right\} \\ &= \left\{v_f + \alpha(v_g - v_f)\right\}\int_f \tau_w\,\mathrm{d}y_f \\ &\quad + \left\{v_g - (1 - \alpha)(v_g - v_f)\right\}\int_g \tau_w\,\mathrm{d}y_g \end{aligned}$$

$$= v_f \int_f \tau_w \, \mathrm{d}y_f + v_g \int \tau_w \, \mathrm{d}y_g$$

$$+ A\alpha(1 - \alpha)(v_g - v_f)\left\{\frac{1}{(1 - \alpha)A}\int_f \tau_w \, \mathrm{d}y_f - \frac{1}{\alpha A}\int \tau_w \, \mathrm{d}y_g\right\}. \tag{25}$$

Combining (24) and (25) yields

$$TS_s = \frac{\mathrm{d}q}{\mathrm{d}t} + v_f \int_f \tau_w \, \mathrm{d}y_f + v_g \int_g \tau_w \, \mathrm{d}y_g + \alpha(1 - \alpha)(v_g - v_f)A\Bigg[g\cos\theta(\rho_f - \rho_g) + \frac{1}{(1 - \alpha)A}\int \tau_w \, \mathrm{d}y_f - \frac{1}{\alpha A}\int \tau_w \, \mathrm{d}y_g + \frac{1}{2}\left\{G(v_g - v_f)\left(\frac{1}{1 - \alpha} - \frac{1}{\alpha}\right)\frac{\mathrm{d}x}{\mathrm{d}z} - \rho_g\frac{\mathrm{d}v_g^2}{\mathrm{d}z} + \rho_f\frac{\mathrm{d}v_f^2}{\mathrm{d}z}\right\}\Bigg]. \tag{26}$$

Equation (26) expresses four mechanisms of entropy generation:

(1) Heat transfer.
(2) Wall shear on the liquid.
(3) Wall shear on the gas.
(4) Relative motion.

The entropy generation due to relative motion is a new phenomenon to those who are used to working with single phase flows and is often overlooked. It is represented by the final term in equation (26) and is zero only if one of the factors in that term is zero. Taking the factors in order we have the following conditions for zero entropy generation due to relative motion

$\alpha = 0$, i.e. single-phase liquid;
$(1 - \alpha) = 0$, i.e. single-phase gas;
$v_g = v_f$, i.e. homogeneous flow.

The only non-trivial result is found if the square bracket is equated to zero and rearranged to give

$$\rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} + g\rho_f\cos\theta + \frac{1}{2}\frac{G(v_g - v_f)}{1 - \alpha}\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{1}{(1 - \alpha)A}\int_f \tau_w \, \mathrm{d}y_f = \rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} + g\rho_g\cos\theta + \frac{1}{2}\frac{G(v_g - v_f)}{\alpha}\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{1}{\alpha A}\int_g \tau_w \, \mathrm{d}y_g. \tag{27}$$

To discover the physical significance of equation (27) we let both sides equal λ, take $(1 - \alpha)$ times the left-hand side and add it to α times the right-hand side. The result is

$$\lambda = (1 - \alpha)\rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} + \alpha\rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} + (v_g - v_f)G\frac{\mathrm{d}x}{\mathrm{d}z} + g\cos\theta\,[(1 - \alpha)\rho_f + \alpha\rho_g]. \tag{28}$$

Comparison with equation (13) and use of (18) reveals the simple result that

$$\lambda = -\frac{\mathrm{d}p}{\mathrm{d}z}. \tag{29}$$

Equation (27) is therefore equivalent to the two equations, which are equations of motion for each phase, as follows

$$\rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} = -\frac{\mathrm{d}p}{\mathrm{d}z} - \rho_f g\cos\theta - \frac{1}{(1 - \alpha)A} \times \int_f \tau_w \, \mathrm{d}y_f - \frac{1}{2}\frac{G(v_g - v_f)}{1 - \alpha}\frac{\mathrm{d}x}{\mathrm{d}z} \tag{30}$$

$$\rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} = -\frac{\mathrm{d}p}{\mathrm{d}z} - \rho_g g\cos\theta - \frac{1}{\alpha A}\int_g \tau_w \, \mathrm{d}y_g - \frac{1}{2}\frac{G(v_g - v_f)}{\alpha}\frac{\mathrm{d}x}{\mathrm{d}z}. \tag{31}$$

Thus it is seen that, in order to conserve entropy, the impulse represented by equation (18) has to be shared equally by the components, each of which experiences a reaction $-\frac{1}{2}\,G(v_g - v_f)\,\mathrm{d}x/\mathrm{d}z$ per unit volume of the duct. Since the terms in the equations of motion represent forces per unit volume of the specific component, this force must be divided by the volumetric concentration of each, as appropriate.

The above equations are equally valid for boiling or condensation and are perfectly symmetrical. At first sight it may appear very strange that the liquid and the vapor should share the impulse which is due to vaporisation. Most authors are wont to attribute all of this impulse to the vapor stream during boiling and to the liquid stream during condensation. The explanation however, should be sought in the two-dimensional effects which have been obscured by the one-dimensional idealizations. In actual fact the local velocities of the liquid and vapor streams tangential to the interface will be equal and there will be velocity variations over the flow field. If solutions could be obtained to the detailed three-dimensional flow problem this would give a more complete representation of "reversible evaporation and condensation".

The conclusion that the impulse represented by equation (18) should be shared equally between the components could also have been derived from consideration of reversibility. Unless the impulse is shared equally, the equations differ in the cases of evaporation and condensation and the fluid cannot be caused to return to its initial state by reversing either process in a symmetrical way.

Colloquially this result could be expressed as follows—"If you are going to use a one-dimensional flow model, and you are going to require that there be no entropy generation, then you must include the final terms in equations (30) and (31) in order to keep all the relevant equations consistent".

Now, suppose that there is irreversible momentum transfer taking place between the vapor and liquid streams. This could be imagined to be the result of a "Reynolds transfer", involving M units of mass per unit volume per unit time, which alternately contact the vapor and liquid and share momentum with each. The resulting force on the liquid will be

$$F'_f = (v_g - v_f)M \tag{32}$$

in the direction of motion. The corresponding reaction on the gas is

$$F'_g = -(v_g - v_f)M. \tag{33}$$

The total force on the liquid per unit volume of the flow field, resulting from the relative motion, assuming that the two processes of reversible and irreversible momentum transfer can be superposed linearly, is

$$F_f = (v_g - v_f)\left[M - \tfrac{1}{2}G\frac{\mathrm{d}x}{\mathrm{d}z}\right] \tag{34}$$

and on the gas, similarly

$$F_g = (v_g - v_f)\left[-M - \tfrac{1}{2}G\frac{\mathrm{d}x}{\mathrm{d}z}\right]. \tag{35}$$

If the interfacial perimeter is P_i in a pipe of area A, then the surface area per unit volume is P_i/A. If ε_o is the irreversible Reynolds mass flux per unit area of the interface then we have

$$\varepsilon_o = \frac{MA}{P_i}. \tag{36}$$

The rate of vaporization per unit area of the interface is

$$m = \frac{GA}{P_i}\frac{\mathrm{d}x}{\mathrm{d}z}. \tag{37}$$

Equations (34) and (35) can then be rewritten as

$$F_f = (v_g - v_f)\frac{P_i}{A}\left[\varepsilon_o - \frac{m}{2}\right] \tag{38}$$

$$F_g = -(v_g - v_f)\frac{P_i}{A}\left[\varepsilon_o + \frac{m}{2}\right]. \tag{39}$$

Dividing equations (38) and (39) by $(1 - \alpha)$ and (α) respectively to obtain the force per unit volume of each component we reach exactly the same result as is given in equations (8) and (9).

The interpretation of these results is interesting from the point of view of irreversible thermodynamics. The term involving "m" involves orderly mass transfer and does not contribute to the entropy production. The Reynolds flux, ε_o, however, is a measure of inherent disorderly mass exchange, irreversibility and entropy production. Use of (10) and (11) in (26), for example, yields the result

$$TS_s = \frac{\mathrm{d}q}{\mathrm{d}t} + v_f \int_f \tau_w \,\mathrm{d}y_f + v_g \int_g \tau_w \,\mathrm{d}y_g + (v_g - v_f)^2 \,\varepsilon_o P_i. \qquad (40)$$

Note that the terms involving wall shear stress in equation (40) are not necessarily positive. In vertical slug flow, for example, the second term can be negative while the third term is zero. The Second Law of Thermodynamics, however, requires that the final term should be large enough to ensure that the net dissipation is positive.

USE OF THE EQUATIONS

The equations which have been developed above are used in much the same way as the equivalent one-dimensional single-phase equations. The continuity, momentum and energy equations are combined with thermodynamic property relationships and solved simultaneously under suitable boundary conditions. Unfortunately, due to the increased complexity of the two-phase flow case, the number of problems which have been solved up to now, and have been tested systematically by means of experiments, is very small. The most that can be done in this paper is to cover the following three topics:

(1) Condensation in straight round ducts with a thin liquid film on the wall.
(2) Comparison with the momentum exchange model of Levy.
(3) The conditions for "choking" in two-phase flow.

(1) *Condensation in straight round ducts with a thin liquid film on the wall*

Let the liquid film have a thickness δ and the pipe have a diameter D. Since the film is thin we can replace α by unity in equation (10), $(1 - \alpha)$ by $4\delta/D$ in equation (11) and neglect the inertia terms in comparison with the viscous terms in equation (11). The ratio P_i/A is equal to $4/D$. Since the liquid presumably covers the wall the value of $\int_g \tau_w \,\mathrm{d}y_g$ is zero. Since condensation is occurring, m_{f_g} is negative and equal to $-m$, say. Furthermore it is usually reasonable at low pressures to assume that $v_g \gg v_f$. With all these assumptions (which should be checked for specific cases) equations (10) and (11) reduce to

$$-\frac{\mathrm{d}p}{\mathrm{d}z} = \rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} + \rho_g g \cos\theta + \frac{4}{D} v_g(\varepsilon_o - m/2). \qquad (41)$$

$$-\frac{\mathrm{d}p}{\mathrm{d}z} = \rho_f g \cos\theta + \frac{\tau_w}{\delta} - \frac{1}{\delta} v_g(\varepsilon_o + m/2). \qquad (42)$$

The continuity equation (14) becomes, with the use of (3) and (4)

$$\pi D m = \frac{\mathrm{d}}{\mathrm{d}z} W_f = -\frac{\mathrm{d}}{\mathrm{d}z}\left[\rho_g v_g \frac{\pi D^2}{4}\right]. \qquad (43)$$

Focusing attention, for the moment, on the vapor, we have on expanding (43)

$$-\frac{4m}{D} = \rho_g \frac{\mathrm{d}v_g}{\mathrm{d}z} + v_g \frac{\mathrm{d}\rho_g}{\mathrm{d}z} + \frac{2}{D}\rho_g v_g \frac{\mathrm{d}D}{\mathrm{d}z}. \qquad (44)$$

Eliminating $\mathrm{d}v_g/\mathrm{d}z$ between (41) and (44) gives

$$-\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{4v_g}{D}\left[\varepsilon_o - \frac{3m}{2} - \tfrac{1}{2}\rho_g v_g \frac{\mathrm{d}D}{\mathrm{d}z}\right] + \rho_g g \cos\theta - v_g^2 \frac{\mathrm{d}\rho_g}{\mathrm{d}z}. \qquad (45)$$

Now, for a two-phase equilibrium mixture, $\mathrm{d}\rho_g/\mathrm{d}z$ can be related to the pressure gradient, as follows

$$\frac{\mathrm{d}\rho_g}{\mathrm{d}z} = \frac{\mathrm{d}p}{\mathrm{d}z}\Big/\frac{\partial p}{\partial \rho_g} = \frac{1}{c_g^2}\frac{\mathrm{d}p}{\mathrm{d}z} \qquad (46)$$

where

$$c_g = \sqrt{\left(\frac{\partial p}{\partial \rho_g}\right)} \qquad (47)$$

is a pseudo-sonic velocity for the vapor in equilibrium with saturated water. Defining a

Mach number for the vapor as follows

$$M_g = v_g/c_g. \tag{48}$$

equation (45) can be rearranged into the form

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\dfrac{4v_g}{D}\left[-\varepsilon_o + \dfrac{3m}{2} + \frac{1}{2}\rho_g v_g \dfrac{\mathrm{d}D}{\mathrm{d}z}\right] - \rho_g g \cos\theta}{1 - M_g^2}. \tag{49}$$

The terms in the square bracket reflect, in order, the effects of friction, condensation and area change. The quantity ε_o is normally determined from a friction factor correlation by using equation (4) of the first paper [1], thus

$$\varepsilon_o = f^{\frac{1}{2}} \rho_g v_g. \tag{50}$$

Defining $G_g = \rho_g v_g$ as the vapor mass flux, an alternative formulation of equation (49) is

$$\frac{\mathrm{d}p}{\mathrm{d}z}(1 - M_g^2) = \frac{2\rho_g v_g^2}{D}\left[\frac{3}{2}\frac{m}{G_g} - f + \frac{\mathrm{d}D}{\mathrm{d}z}\right] - \rho_g g \cos\theta. \tag{51}$$

Typical values for G_g under condensing conditions [e.g. steam at 500 ft/s at 90°F] are about 4000 lb/h ft^2 or less, f is about 0·005 and $\mathrm{d}D/\mathrm{d}z$ is usually zero but could conceivably be made as high as 0·1. A high value of m is 40 lb/h ft^2 which gives the first term in the bracket a value of 0·03. On the other hand, for a net condensing heat-transfer coefficient of 1000 Btu/h ft^2 °F and a temperature difference of 10 degF for steam, m is only 10 lb/h ft^2 and the first term becomes 0·0075. Thus slight pressure recovery is possible in a constant area duct but could be made far greater by using a diverging diffuser since the last term in the square bracket could be about one hundred times as large as the sum of the other terms. Condensation on the walls of such a diffuser would act like boundary-layer suction and would tend to prevent separation.

This theory has been partially tested experimentally both in an analog apparatus, in which air was extracted through the walls of a porous tube [4] and in a vertical condenser [5]. In both cases the Mach number was less than 0·2, the duct diameter was constant and the gravitational term was negligible. For a constant rate of condensation (or extraction) and small density changes down the duct, equation (51) can be integrated over a length L to give the total pressure rise in terms of the inlet and outlet mean vapor velocities $(V_g)_1$ and $(V_g)_2$. The result is

$$\frac{P_i - P_i}{\frac{1}{2}\rho_g(v_g)_1^2} = 1{\cdot}5\left[1 - \left\{\frac{(V_g)_2}{(V_g)_1}\right\}^2\right] - \frac{4}{3} f \frac{L}{D}\left\{1 + \frac{(V_g)_2}{(V_g)_1} + \frac{(V_g)_2^2}{(V_g)_1^1}\right\}. \tag{52}$$

The friction factor is assumed to be given by the equivalent value for the vapor flowing alone in the duct.

The data in both the analog and the real condenser correlated with an equation of the form of (52) except that the constant in front of the first bracket was found to be 1·7 instead of 1·5. This is in contrast to the result which would have been achieved using the assumption that the value of ε_m is unaffected by condensation and is always equal to ε_o. In this latter case the value of the constant is unity.

Some further insight into these effects is obtained by rewriting the first term on the right-hand side of (52) to give

$$\frac{P_2 - P_1}{\frac{1}{2}\rho_g(v_g)_1^2} = (1 + C)\left[1 - \left\{\frac{(V_g)_2}{(V_g)_1}\right\}^2\right] - \frac{4}{3} f \frac{L}{D}\left[1 + \frac{(V_g)_2}{(V_g)_1} + \frac{(V_g)_2^2}{(V_g)_1^2}\right]. \tag{53}$$

This equation is compared with experimental data in Figs. 2 and 3. In the simplest one-dimensional model $C = 0{\cdot}5$. However, it can differ from this value due to two effects:

(1) An asymmetrical turbulence pattern, or a quiescent region on the wall, which introduces the factor β in equation (18) of the first paper.

(2) Velocity profile effects. The mean of the

velocity squared across a duct, is in general, greater than the square of the mean velocity. Therefore the one-dimensional assumption underestimates the momentum flux. The actual mean momentum flux is then $\gamma\rho V^2$ where γ is close to unity for smooth-walled pipes but may be as high as 1·1 in rough pipes.

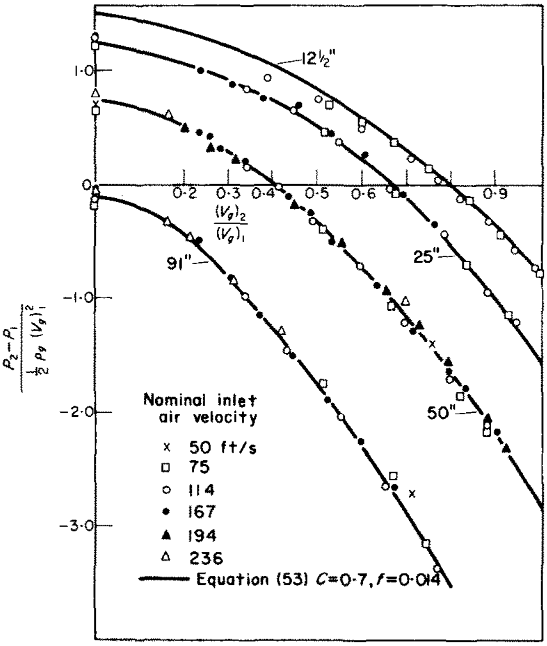

FIG. 2. Dimensionless pressure change vs. velocity ratio for various tube lengths and velocities (Wallis, porous tube).

Combing these effects it is readily shown by an overall momentum balance between sections (1) and (2) that

$$C = 2\gamma - 2 + \beta. \tag{54}$$

In Wallis' paper [4] the value of γ was determined, by means of a pilot traverse, to be equal to 1·07. With the observed value of $C = 1{\cdot}7$, therefore, the empirical value of β is 0·56.

An interesting prediction of equation (53) is that the theoretical pressure recovery in a very short tube is $(1 + C)$ times the value of $\frac{1}{2}\rho_g v_g^2$. This can be explained in terms of two-dimensional effects. The maximum pressure recovery is obtained by reversibly diffusing the center streamline. This streamline has a velocity about 1·2 to 1·3 times the average and a dynamic head of about 1·45 and 1·75 times the average. On this basis alone one would therefore expect values of C between 0·45 and 0·75. In Wallis' experiments [4] in a rough pipe the maximum velocity head was observed to be 1·75 times the average, which is in good agreement with the observed value of $C = 0{\cdot}7$.

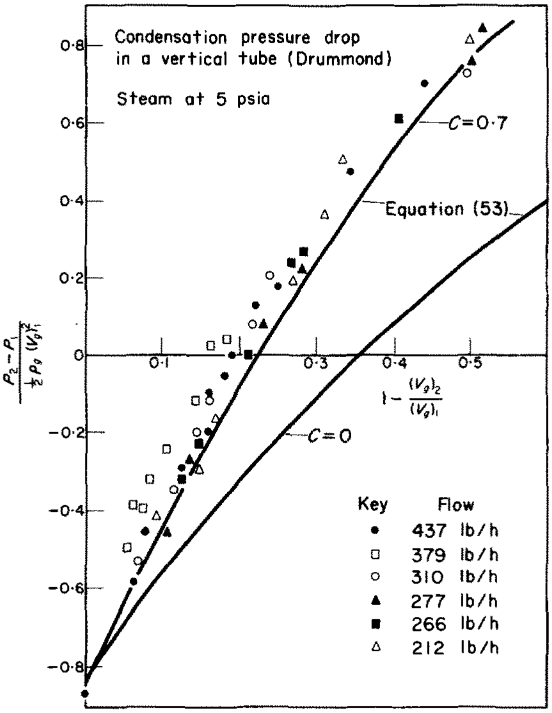

FIG. 3. Condensation pressure drop in a vertical tube (Drummond).

We note at this point that high rates of condensation or extraction would correspond to values of m greater than $2\varepsilon_0$. If equation (28) of the first paper is used in this case there would be no contribution to the equation of motion of the vapor due to condensation or interfacial shear. The whole of the last term would vanish from equation (41) and the result of carrying through the complete analysis would be, instead of equation (52) the following,

$$\frac{P_2 - P_1}{\frac{1}{2}\rho_g (v_g)_1^2} = 1 - \left\{\frac{(v_g)_2}{(v_g)_1}\right\}^2. \tag{55}$$

However, if equation (27) of the first paper is retained *even for values of m greater than* $2\varepsilon_o$, equation (52) remains unchanged and the theory predicts 50 per cent more pressure recovery in short tubes, in better agreement with experimental results. Thus it appears quite valid to allow ε_m to take on negative values. There is no violation of the Second Law of Thermodynamics in this case because we have already shown that equation (7) is consistent with a mechanism of entropy production which depends only on ε_o, even when ε_m happens to be negative.

An interesting twist to this argument is the following: Starting from a one-dimensional model and considering only the methods of entropy generation we arrive at equation (30). Applying this equation to the problem of rapid condensation in straight ducts we find that the pressure recovery is 1·5 times what we expected. This indicates that the central streamline must have a dynamic head of 1·5 times the average. This "deduction" can be made without any *a priori* knowledge of turbulent flow velocity profiles.

The liquid film. Equation (42) provides an overall force balance for the liquid film in which the pressure gradient is balanced by gravitational forces, wall shear stress and interfacial shear stress. A "lumped-parameter" description can then be obtained by relating the wall shear stress to the liquid average velocity by means of a friction factor, as follows,

$$\tau_w = f_w \cdot \tfrac{1}{2}\rho_f v_f^2. \tag{56}$$

From the continuity equation, for a thin film

$$W_f = \pi D \delta v_f \rho_f. \tag{57}$$

Combining (56) and (57) with (42) we get

$$\rho_f g \cos\theta + \frac{\mathrm{d}p}{\mathrm{d}z} + f_w \cdot \frac{1}{2\rho_f}\left(\frac{W_f}{\pi D}\right)^2 \frac{1}{\delta^3} - \frac{v_g}{\delta}\left(\varepsilon_o + \frac{m}{2}\right) = 0. \tag{58}$$

If m is known and $\mathrm{d}p/\mathrm{d}z$ is calculated using (51), this equation can then be solved for δ. In the case of turbulent films a reasonable assumption is that $f = 0{\cdot}008$ for values of the film Reynolds number, $Re_f = 4W_f/\pi D\mu_f$ greater than about 2000. In the case of laminar films it is better to solve the differential equations for the velocity profile in the film to obtain, eventually, the result

$$\left(\rho_f g \cos\theta + \frac{\mathrm{d}p}{\mathrm{d}z}\right)\frac{\delta^3}{3\mu_f} - \frac{\delta^2}{2\mu_f} v_g \left(\varepsilon_o + \frac{m}{2}\right) + \frac{W_f}{\pi D \rho_f} = 0 \tag{59}$$

which is almost the same as assuming that the wall friction factor is equal to $16/Re_f$.

In order to solve the whole problem iteration is usually necessary since the film thickness has a direct influence on the condensing rate. A smaller film thickness increases the rate of condensation, and therefore the interfacial shear, which itself decreases the film thickness still further.

In cases where θ is not $\pm 90°$ it may be necessary to worry about the assymetrical distribution of the liquid film around the duct circumference. Furthermore, if the film thickness is large, or vapor velocities are high, waves will be generated and the value of ε_0 will increase. Under these conditions use can perhaps be made of two-phase flow correlations in the literature (e.g. [6, 7]).

Indeed, a test of the present theory against the data of Goodykoontz and Dorsch [11] revealed that it was necessary to increase ε_o by about a factor of two in order to get agreement with the data. This symptom of an increased friction factor occurred even at the inlet end of the condenser where the liquid film thickness was about 10^{-4} in. Perhaps this was due to film instability as a result of the very high steam velocities (500–1000 ft/s). Evidently some further work would be worthwhile to determine just when a modification to ε_o is necessary to account for film waviness and how the analysis should then be modified.

2. *Derivation of Levy's "momentum exchange model"*

The pressure gradient can be eliminated by subtracting equation (6) from equation (5), as follows:

$$\rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} - \rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} = (\rho_f - \rho_g)\, g \cos\theta + \frac{1}{A}\left[\frac{1}{1-\alpha}\int_f \tau_w \,\mathrm{d}y_f - \frac{1}{\alpha}\int_g \tau_w \,\mathrm{d}y_g\right] - \frac{P_i}{A}(v_g - v_f)\left[\varepsilon_m\left(\frac{1}{1-\alpha} + \frac{1}{\alpha}\right) + \frac{m}{\alpha}\right]. \qquad (60)$$

Levy [8] argued that, since $\mathrm{d}v_g/\mathrm{d}z$, $\mathrm{d}v_f/\mathrm{d}z$ and m are all zero in the particular case in which there is no acceleration or heat addition, the sum of all the other terms in the equation could be taken to be approximately zero at all times, thus leaving the relation

$$\rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} - \rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} + \frac{P_i}{A}(v_g - v_f)\frac{m}{\alpha} = 0. \qquad (61)$$

Using (18) this becomes

$$\rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} - \rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} + \frac{1}{\alpha}(v_g - v_f)\, G \frac{\mathrm{d}x}{\mathrm{d}z} = 0 \qquad (62)$$

from which all of Levy's results can be obtained.

An alternative derivation of equation (61) from equation (60) can be based on the elimination of the unwanted terms by assuming a sufficiently rapid rate of evaporation and expansion in which case the gravitational and frictional terms become negligible compared with the inertia terms and the momentum exchange terms due to vaporization. Furthermore, by defining "rapid evaporation" by the criterion, $m > 2\varepsilon_o$, we can use equation (28) of the previous paper to eliminate the term in ε_m. This condition of "rapid evaporation" is described in Levy's paper [8] in terms of low values of a parameter "K" and there appears to be a distinct improvement in the accuracy of the model as K is reduced.

The particular value of equation (62) is that it can be used to form a perfect differential in terms of x and α as long as G and ρ_f are kept constant. By substituting from (3, 4, 15–17) it is readily found that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{(1-x)^2}{1-\alpha} + \frac{x^2}{\alpha}\frac{\rho_f}{\rho_g} - \frac{1}{2}\frac{(1-x)^2}{(1-\alpha)^2}\right] = 0. \qquad (63)$$

By integrating equation (63) along the duct an expression relating void fraction and quality can be obtained. This closed form solution is sufficiently simple to be valuable even when the assumptions which form the basis of the model are not rigorously valid.

It should be noted that the momentum exchange model is not compatible with the conditions for isentropic expansion. In order to achieve such a condition under the one-dimensional flow assumptions we should have to use equation (7) and equate ε_o to zero rather than ε_m. The resulting equation in place of (62) is then

$$\rho_g v_g \frac{\mathrm{d}v_g}{\mathrm{d}z} - \rho_f v_f \frac{\mathrm{d}v_f}{\mathrm{d}z} + (v_g - v_f)\frac{G}{2}\left[\frac{1}{\alpha} - \frac{1}{1-\alpha}\right]\frac{\mathrm{d}x}{\mathrm{d}z} = 0. \qquad (64)$$

Unfortunately, equation (64) does not appear to give rise to a complete differential such as equation (63) and is therefore less easy to use in practice.

3. *Criteria for "choking" in two-phase flow*

We commence by processing the momentum and continuity equations of each phase in a way which is analogous to the treatment which is standard for single phase flows. Firstly, we use the relation

$$v_g \rho_g \frac{\mathrm{d}v_g}{\mathrm{d}z} = v_g \frac{\mathrm{d}(\rho_g v_g)}{\mathrm{d}z} - v_g^2 \frac{\mathrm{d}\rho_g}{\mathrm{d}z}. \qquad (65)$$

Now, in the two-phase region at equilibrium,

ρ_g is determined uniquely by the pressure and we may write

$$\frac{\mathrm{d}\rho_g}{\mathrm{d}z} = \frac{\mathrm{d}p}{\mathrm{d}z}\bigg/\frac{\partial p}{\partial \rho_g}. \tag{66}$$

Define a pseudo-sonic velocity for the vapor as

$$c_g^2 = \left(\frac{\partial p}{\partial \rho_g}\right). \tag{67}$$

Substituting (65) into (5) and using (66) and (67) we obtain

$$-\frac{\mathrm{d}p}{\mathrm{d}z}\left[1 - \frac{v_g^2}{c_g^2}\right] = v_g\frac{\mathrm{d}(\rho_g v_g)}{\mathrm{d}z} + \rho_g g\cos\theta + \frac{1}{\alpha A}\int \tau_w\,\mathrm{d}y_g + \frac{P_i}{A\alpha}(v_g - v_f)(\varepsilon_m + m). \tag{68}$$

Equations (3) and (15) are combined to give

$$\rho_g v_g \frac{\alpha}{x} A = W. \tag{69}$$

Taking the logarithm of both sides of (69) and differentiating we get

$$\frac{1}{\rho_g v_g}\frac{\mathrm{d}(\rho_g v_g)}{\mathrm{d}z} + \frac{1}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}z} + \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}z} - \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}z} = 0. \tag{70}$$

Combining (68) and (70) by the elimination of $\mathrm{d}(\rho_g v_g)/\mathrm{d}z$ yields

$$-\frac{\mathrm{d}p/\mathrm{d}z}{\rho_g v_g^2}\left[1 - \frac{v_g^2}{c_g^2}\right] = -\frac{1}{\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}z} + \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{\rho_g g\cos\theta + (1/\alpha A)\int_g \tau_w\,\mathrm{d}y_g}{\rho_g v_g^2} + \frac{P_i}{A\alpha}\frac{(v_g - v_f)}{\rho_g v_g^2}(\varepsilon_m + m). \tag{71}$$

The similar equation for the liquid is

$$-\frac{\mathrm{d}p/\mathrm{d}z}{\rho_f v_f^2}\left[1 - \frac{v_f^2}{c_f^2}\right] = \frac{1}{1-\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}z} - \frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}z} - \frac{1}{1-x}\frac{\mathrm{d}x}{\mathrm{d}z} + \frac{\rho_f g\cos\theta + \frac{1}{(1-\alpha)A}\int_f \tau_w\,\mathrm{d}y_f}{\rho_f v_f^2} - \frac{P_i}{A(1-\alpha)}\frac{(v_g - v_f)}{\rho_f v_f^2}\varepsilon_m. \tag{72}$$

Equations (71) and (72) resemble the general one-dimensional equations of single component flow, except for the effects of phase change and the additional degree of geometrical freedom which is introduced by the variable α. "Choking" occurs in the vapor phase when $v_g = c_g$, but this does not necessarily correspond to the "compound-choking" of the combined flows since α is free to adjust to the local conditions.

In order to investigate "compound-choking" we eliminate $\mathrm{d}\alpha/\mathrm{d}z$ between equations (71) and (72). Wall shear stress and gravitational terms are also dropped for simplicity and because they are negligible near the choking condition where the pressure gradient becomes very large. The result is

$$-\frac{\mathrm{d}p}{\mathrm{d}z}\left[\frac{\alpha}{\rho_g v_g^2}\left(1 - \frac{v_g^2}{c_g^2}\right) + \frac{(1-\alpha)}{\rho_f v_f^2}\left(1 - \frac{v_f^2}{c_f^2}\right)\right] = -\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}z} + \frac{\mathrm{d}x}{\mathrm{d}z}\left[\frac{\alpha}{x} - \frac{1-\alpha}{1-x}\right] + \frac{P_i}{A}(v_g - v_f)\left[\frac{\varepsilon_m + m}{\rho_g v_g^2} - \frac{\varepsilon_m}{\rho_f v_f^2}\right]. \tag{73}$$

As long as $\mathrm{d}x/\mathrm{d}z$, m and ε_m are independent of the pressure gradient, the choking condition is

$$\frac{\alpha}{\rho_g v_g^2}\left(1 - \frac{v_g^2}{c_g^2}\right) + \frac{1-\alpha}{\rho_f v_f^2}\left(1 - \frac{v_f^2}{c_f^2}\right) = 0. \tag{74}$$

However, the presence of flashing, which is the formation of vapor due to a change in pressure, introduces a dependence on pressure gradient in the terms on the right-hand side of equation (73). Quite generally we may say that the quality, x, is a function of pressure and some other thermodynamic property, a. The rate of change of quality is then

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \left(\frac{\partial x}{\partial p}\right)_a\frac{\mathrm{d}p}{\mathrm{d}z} + \left(\frac{\partial x}{\partial a}\right)_p\frac{\mathrm{d}a}{\mathrm{d}z}. \tag{75}$$

If we choose the property "a" to be the one which is invariant during the thermodynamic path which is followed by the flow, we have

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \left(\frac{\partial x}{\partial p}\right)_a \frac{\mathrm{d}p}{\mathrm{d}z}. \tag{76}$$

Furthermore, m may be related to $\mathrm{d}x/\mathrm{d}z$ by the use of equation (18), ε_m can be expressed in terms of m using relations such as equation (7), and eventually all the terms involving the pressure gradient can be gathered together to give a criterion for choking in the presence of flashing. Of course, any non-equilibrium effects will complicate the situation.

If equation (7) is used the criterion for choking is [using (3, 4, 15–17)]

$$\frac{\alpha}{\rho_g v_g^2}\left(1 - \frac{v_g^2}{c_g^2}\right) + \frac{1-\alpha}{\rho_f v_f^2}\left(1 - \frac{v_f^2}{c_f^2}\right) + \left(\frac{\partial x}{\partial p}\right)_a \left[\frac{3 - v_f/v_g}{\rho_g v_g} + \frac{v_g/v_f - 3}{\rho_f v_f}\right]\frac{G}{2} = 0. \tag{77}$$

Since equation (7) is compatible with the constant entropy assumption, the property "a" should therefore be taken to be the entropy.

On the other hand, if the assumptions of the momentum exchange model are used, ε_m is independent of m and the result is

$$\frac{\alpha}{\rho_g v_g^2}\left(1 - \frac{v_g^2}{c_g^2}\right) + \frac{1-\alpha}{\rho_f v_f^2}\left(1 - \frac{v_f^2}{c_f^2}\right) + \left(\frac{\partial x}{\partial p}\right)_a \left[\frac{2v_g - v_f}{\rho_g v_g^2} - \frac{1}{\rho_f v_f}\right] G = 0. \tag{78}$$

In this case the thermodynamic path cannot be taken to be exactly a constant entropy process and the value of "a" remains to be determined. Fortunately, in the case of a steam–water mixture expanding without heat addition, the path lies between the two extremes of constant enthalpy and constant entropy and the result is not very dependent on which process is assumed.

The method used by Levy [9] to correlate the choking flow of steam–water mixtures is equivalent to equation (78) and gives reasonable agreement with the data. The corresponding result from equation (77) is not so easy to compute because of the difficulty of relating quality and void fraction explicitly. An interesting development, however, is to examine under what conditions the result will be *independent of the assumptions which are made about* ε_m. From (73) we see that this is the case if

$$\rho_g v_g^2 = \rho_f v_f^2 \tag{79}$$

which is identical with the assumptions made by Fauske [10].* In this case, the isentropic model, the momentum exchange model and Fauske's model (indeed, all models) give exactly the same result. Moreover, the equations are simplified considerably. The use of (79) in any of equations (73, 77) or (78) gives

$$(1-\alpha)\frac{v_f^2}{c_f^2} + \alpha\frac{v_g^2}{c_g^2} - 2G(v_g - v_f)\left(\frac{\partial x}{\partial p}\right)_a = 1. \tag{80}$$

Furthermore, as a result of (79), the void fraction can be expressed directly in terms of the quality, as follows

$$\alpha = \frac{1}{1 + [(1-x)/x]\sqrt{(\rho_g/\rho_f)}}. \tag{81}$$

Usually $c_f^2 \gg c_g^2$, $\rho_f \gg \rho_g$ and the first term in (80) can be neglected in comparison with the second. Except at high pressures, moreover, $v_g \gg v_f$ in the last term of (80) and the very simple result follows

$$\alpha\frac{v_g^2}{c_g^2} - 2Gv_g\left(\frac{\partial x}{\partial p}\right)_a = 1. \tag{82}$$

Expressing v_g in terms of G we find

$$G^2\left[\frac{x^2}{\rho_g^2 \alpha c_g^2} - 2\left(\frac{\partial x}{\partial p}\right)_a \frac{x}{\alpha\rho_g}\right] = 1. \tag{83}$$

Since $\partial x/\partial p$ *for water* at low qualities is not very sensitive to the thermodynamic path we may choose "a" to be the entropy for convenience.

The flow rate per unit area at the choking

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* Equation (79) is not to be regarded as giving a good approximation to the velocity ratio. It is merely a mathematical expression of the condition under which all of the models will give the same answer.

condition is obtained from (83) to be

$$G_{\text{choking}} = \left[\frac{x^2}{\rho_g^2 \alpha c_g^2} - 2\left(\frac{\partial x}{\partial p}\right)_s \frac{x}{\alpha \rho_g}\right]^{-\frac{1}{2}}. \quad (84)$$

The first term in the bracket dominates for water at large values of x, whereas at very low qualities the second term dominates. At low pressures and high qualities $\alpha \approx 1$ and the equation reduces to

$$G \approx \frac{\rho_g c_g}{x} \quad (85)$$

which is equivalent to assuming that choking of the vapor alone occurs.

Equation (84) was solved simultaneously with (81) for water in the range $x = 0{\cdot}03$ to $x = 0{\cdot}8$, $p = 15$–300 psia and the results were found to be indistinguishable from the results of Levy reported in [9]. The equations in this form are much easier to use than Levy's calculation procedure.

Summary

It has been shown that several "models" for two-phase choking (critical flow) can be deduced by making appropriate assumptions about the Reynolds flux with phase change, ε_m. Thus this parameter provides a unifying concept to which the various theories can be related and is evidently of utility for purposes of analysis. In the case of steam–water flow the results are singularly unaffected by the choice of assumptions about ε_m.

The previous analysis only applies to flows in which thermodynamic equilibrium between the liquid and the vapor can be assumed. However, the Reynolds flux concept can presumably still be applied to non-equilibrium flows as long as the true rate of vapor generation at the interface can be predicted.

CONCLUSIONS

The utility of the Reynolds flux concept has been demonstrated in several areas of one-dimensional two-phase flow analysis. Further developments are anticipated by using the general equations in other contexts. Although by no means exact, the analytical approach gives usefully accurate predictions without enormous effort, and is a useful tool for exploring a field which all too often appears hopelessly complex.

Specifically, the theory has been shown to give (1) improved prediction of the pressure gradient during film condensation, (2) a description of the mechanics of entropy production in one-dimensional two-phase flow, (3) a unifying concept relating various analytical models for one-dimensional compressible two-phase flow in particular (4) the criteria for "choking".

REFERENCES

1. G. B. Wallis, Use of the Reynolds flux concept for analyzing one-dimensional two-phase flow. Part I. Derivation and verification of basic analytical techniques, *Int. J. Heat Mass Transfer* **11**, 445 (1968).
2. N. Zuber, On the stability of boiling heat transfer, *Trans. Am. Soc. Mech. Engrs* **80**, 711 (1958).
3. S. S. Kutateladze and A. I. Leontiev, Some applications of the asymptotic theory of the turbulent boundary layer, in *Proceedings of the Third International Heat Transfer Conference*, Paper No. 81, Vol. 5, pp. 1–6. Am. Inst. Chem. Engrs, New York (1966).
4. G. B. Wallis, Pressure gradients for air flowing along porous tubes with uniform extraction at the walls, *Proc. Instn Mech. Engrs* **180**, 27–35 (1965–66).
5. G. Drummond, Ph.D. thesis, Heriot–Watt University, Edinburgh, Scotland (1966)
6. O. P. Bergelin, P. K. Kegel, F. G. Carpenter and C. Gazley, Heat Transfer and Fluid Mechanics Inst., Berkeley, Calif. (1949). See W. M. Rohsenow and H. Y. Choi, *Heat, Mass and Momentum Transfer* p. 246. Prentice-Hall, Englewood Cliffs, N.J. (1961).
7. R. W. Lockhart and R. C. Martinelli, Proposed correlation of data for isothermal two-phase, two-component flow in pipes, *Chem. Engng Prog.* **45**, 39 (1949).
8. S. Levy, Steam slip—theoretical prediction from momentum model, *J. Heat Transfer* **82**, 113–124 (1960).
9. S. Levy, Prediction of two-phase critical flow rate, *J. Heat Transfer* **87**, 53–58 (1965).
10. H. K. Fauske, The discharge of saturated water through tubes, AIChE Preprint No. 30, Seventh National Heat Transfer Conference, Cleveland, Ohio (1964).
11. J. H. Goodykoontz and R. G. Dorsch, Local heat transfer coefficients and static pressures for condensation of high-velocity steam within a tube, NASA TN D-3953 (May 1967).